

One-Generator Quasi-Abelian Codes Revisited*

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Abstract

The class of 1-generator quasi-abelian codes over finite fields is revisited. Alternative and explicit characterization and enumeration of such codes are given. An algorithm to find all 1-generator quasi-abelian codes is provided. Two 1-generator quasi-abelian codes whose minimum distances are improved from Grassl's online table are presented.

Keywords. group algebras, quasi-abelian codes, minimum distances, 1-generator.

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1 Introduction

As a family of codes with good parameters, rich algebraic structures, and wide ranges of applications (see [9], [11], [13], [12], [15], [18], and references therein), quasi-cyclic codes have been studied for a half-century. Quasi-abelian codes, a generalization of quasi-cyclic codes, have been introduced in [19] and extensively studied in [8].

Given finite abelian groups $H \leq G$ and a finite field \mathbb{F}_q , an H -quasi-abelian code is defined to be an $\mathbb{F}_q[H]$ -submodule of $\mathbb{F}_q[G]$. Note that H -quasi-abelian codes are not only a generalization of quasi-cyclic codes (see [9], [11], [8], and [19]) if H is cyclic but also of abelian codes (see [1] and [2]) if $G = H$, and of cyclic codes (see [14]) if $G = H$ is cyclic. The characterization and enumeration of quasi-abelian codes have been established in [8]. An H -quasi-abelian code C is said to be of 1-generator if C is a cyclic $\mathbb{F}_q[H]$ -module. Such a code can be viewed as a generalization of 1-generator quasi-cyclic codes which are more frequently studied and applied (see [15], [18], and [13]). Analogous to the case of 1-generator quasi-cyclic codes, the number of 1-generator quasi-abelian codes has been determined in [8]. However, an explicit construction and an algorithm to determine all 1-generator quasi-abelian codes have not been well studied.

In this paper, we give an alternative discussion on the algebraic structure of 1-generator quasi-abelian codes and an algorithm to find all 1-generator quasi-abelian codes. Examples of new codes derived from 1-generator quasi-abelian codes are presented.

The paper is organized as follows. In Section 2, we recall some notations and basic results. An alternative discussion on the algebraic structure of 1-generator quasi-abelian

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codes is given in Section 3 together with an algorithm to find all 1-generator quasi-abelian codes and the number of such codes. Examples of new codes derived from 1-generator quasi-abelian codes are presented in Section 4.

2 Preliminaries

Let \mathbb{F}_q denote a finite field of order q and let G be a finite abelian group of order n , written additively. Denote by $\mathbb{F}_q[G]$ the *group ring* of G over \mathbb{F}_q . The elements in $\mathbb{F}_q[G]$ will be written as $\sum_{g \in G} \alpha_g Y^g$, where $\alpha_g \in \mathbb{F}_q$. The addition and the multiplication in $\mathbb{F}_q[G]$ are given as in the usual polynomial rings over \mathbb{F}_q with the indeterminate Y , where the indices are computed additively in G . We note that $Y^0 = 1$ is the identity of $\mathbb{F}_q[G]$, where 1 is the identity in \mathbb{F}_q and 0 is the identity of G .

Given a ring \mathcal{R} , a linear code of length n over \mathcal{R} refers to a submodule of the \mathcal{R} -module \mathcal{R}^n . A *linear code* in $\mathbb{F}_q[G]$ refers to an \mathbb{F}_q -subspace C of $\mathbb{F}_q[G]$. This can be viewed as a linear code of length n over \mathbb{F}_q by indexing the n -tuples by the elements in G . The *Hamming weight* $\text{wt}(\mathbf{u})$ of $\mathbf{u} = \sum_{g \in G} u_g Y^g \in \mathbb{F}_q[G]$ is defined to be the number of nonzero term u_g 's in \mathbf{u} . The *minimum Hamming distance* a code C is defined by $d(C) := \min\{\text{wt}(\mathbf{u}) \mid \mathbf{u} \in C, \mathbf{u} \neq 0\}$. A code C is referred to as an $[n, k, d]_q$ code if C has \mathbb{F}_q -dimension k and minimum Hamming distance d .

Given a subgroup H of G , a code C in $\mathbb{F}_q[G]$ is called an *H-quasi-abelian code* if C is an $\mathbb{F}_q[H]$ -module, i.e., C is closed under the multiplication by the elements in $\mathbb{F}_q[H]$. Such a code will be called a *quasi-abelian code* if H is not specified or where it is clear in the context. An H -quasi-abelian code C is said to be of *1-generator* if C is a cyclic $\mathbb{F}_q[H]$ -module.

Assume that $H \leq G$ such that $|H| = m$ and the index $[G : H] = \frac{n}{m} = l$. Let $\{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_l\}$ be a fixed set of representatives of the cosets of H in G . Let $R := \mathbb{F}_q[H]$. Define $\Phi : \mathbb{F}_q[G] \rightarrow R^l$ by

$$\Phi\left(\sum_{h \in H} \sum_{i=1}^l \alpha_{h+\mathbf{g}_i} Y^{h+\mathbf{g}_i}\right) = (\alpha_1(Y), \alpha_2(Y), \dots, \alpha_l(Y)), \quad (2.1)$$

where $\alpha_i(Y) = \sum_{h \in H} \alpha_{h+\mathbf{g}_i} Y^h \in R$, for all $i = 1, 2, \dots, l$. It is not difficult to see that Φ is an R -module isomorphism, and hence, the next lemma follows.

Lemma 2.1. *The map Φ induces a one-to-one correspondence between H -quasi-abelian codes in $\mathbb{F}_q[G]$ and linear codes of length l over R .*

Throughout, assume that $\gcd(q, |H|) = 1$, or equivalently, $\mathbb{F}_q[H]$ is semisimple. Following [8, Section 3], the group ring $R = \mathbb{F}_q[H]$ is decomposed as follows.

For each $h \in H$, denote by $\text{ord}(h)$ the order of h in H . A *q-cyclotomic class* of H containing $h \in H$, denoted by $S_q(h)$, is defined to be the set

$$S_q(h) := \{q^i \cdot h \mid i = 0, 1, \dots\} = \{q^i \cdot h \mid 0 \leq i \leq \nu_h\},$$

where $q^i \cdot h := \sum_{j=1}^{q^i} h$ in H and ν_h is the multiplicative order of q in $\mathbb{Z}_{\text{ord}(h)}$.

An *idempotent* in a ring R is a non-zero element e such that $e^2 = e$. An idempotent e is said to be *primitive* if for every other idempotent f , either $ef = e$ or $ef = 0$. The primitive idempotents in R are induced by the q -cyclotomic classes of H (see [5, Proposition II.4]).

Every idempotent e in R can be viewed as a unique sum of primitive idempotents in R . The \mathbb{F}_q -dimension of an idempotent $e \in R$ is defined to be the \mathbb{F}_q -dimension of Re .

Form [8, Subsection 3.2], $R := \mathbb{F}_q[H]$ can be decomposed as

$$R = Re_1 + Re_2 + \cdots + Re_s,$$

where e_1, e_2, \dots, e_s are the primitive idempotents in R . Moreover, every ideal in R is of the form Re , where e is an idempotent in R .

3 1-Generator Quasi-Abelian Codes

In [8], characterization and enumeration of 1-generator H -quasi-abelian codes in $\mathbb{F}_q[G]$. In this section, we give alternative characterization and enumeration of such codes. The characterization in Subsection 3.1 allows us to express an algorithm to find all 1-generator H -quasi-abelian codes in $\mathbb{F}_q[G]$ in Subsection 3.2.

Using the R -module isomorphism Φ defined in (2.1), to study 1-generator H -quasi-abelian codes in $\mathbb{F}_q[G]$, it suffices to consider cyclic R -submodules $R\mathbf{a}$, where $\mathbf{a} = (a_1, a_2, \dots, a_l) \in R^l$.

For each $\mathbf{a} = (a_1, a_2, \dots, a_l) \in R^l$, there exists a unique idempotent $e \in R$ such that $Re = Ra_1 + Ra_2 + \cdots + Ra_l$. The element e is called an *idempotent generator element* for $R\mathbf{a}$. An idempotent $f \in R$ of largest \mathbb{F}_q -dimension such that

$$f\mathbf{a} = 0$$

is called an *idempotent check element* for $R\mathbf{a}$.

Let $S = \mathbb{F}_{q^l}[H]$, where \mathbb{F}_{q^l} is an extension field of \mathbb{F}_q of degree l . Let $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ be a fixed basis of \mathbb{F}_{q^l} over \mathbb{F}_q . Let $\varphi : R^l \rightarrow S$ be an R -module isomorphism defined by

$$\mathbf{a} = (a_1, a_2, \dots, a_l) \mapsto A = \sum_{i=1}^l \alpha_i a_i.$$

Using the map φ , the code $R\mathbf{a}$ can be regarded as an R -module RA in S .

Lemma 3.1 ([8, Lemma 6.1]). *If e and f are idempotent generator and idempotent check elements of $R\mathbf{a}$, respectively, then*

$$e + f = 1$$

and

$$\dim_{\mathbb{F}_q}(R\mathbf{a}) = \dim_{\mathbb{F}_q}(Re) = m - \dim_{\mathbb{F}_q}(Rf).$$

For a ring \mathcal{R} , denote by \mathcal{R}^* and \mathcal{R}^\times the set of non-zero elements and the group of units of \mathcal{R} , respectively.

In order to enumerate and determine all 1-generator H -quasi-abelian codes in $\mathbb{F}_q[G]$, we need the following results.

Lemma 3.2. *Let e be the idempotent generator of $R\mathbf{a}$ and let $\mathbf{b} \in R^l$. Let $A = \varphi(\mathbf{a})$ and $B = \varphi(\mathbf{b})$. Then $R\mathbf{a} = R\mathbf{b}$ if and only if there exists $u \in (Re)^\times$ such that $\mathbf{b} = u\mathbf{a}$.*

Equivalently, $RA = RB$ if and only if there exists $u \in (Re)^\times$ such that $B = uA$.

Proof. Assume that $R\mathbf{a} = R\mathbf{b}$. Then $\mathbf{b} = v\mathbf{a}$ for some $v \in R$. Let $u = ve \in Re$. Note that $a_i = r_i e$, where $r_i \in R$, for all $i = 1, 2, \dots, l$. Then $ua_i = ver_i e = v(r_i e) = va_i = b_i$ for all $i = 1, 2, \dots, l$. Hence, $\mathbf{b} = u\mathbf{a}$ and

$$Re = Rb_1 + Rb_2 + \dots + Rb_l = v(Ra_1 + Ra_2 + \dots + Ra_l) = vRe = (ve)Re = uRe.$$

Since $u \in Re$ and $Re = uRe$, we have $u \in (Re)^\times$.

Conversely, assume that there exists $u \in (Re)^\times$ such that $\mathbf{b} = u\mathbf{a}$. Then $R\mathbf{b} = Ru\mathbf{a} \subseteq R\mathbf{a}$. We need to show that $\dim_{\mathbb{F}_q}(R\mathbf{a}) = \dim_{\mathbb{F}_q}(R\mathbf{b})$. Let e' be an idempotent generator of $R\mathbf{b}$. We have

$$\begin{aligned} Re' &= Rb_1 + Rb_2 + \dots + Rb_l \\ &= R(ua_1) + R(ua_2) + \dots + R(ua_l) \\ &= u(Ra_1 + Ra_2 + \dots + Ra_l) \\ &= u(Re) = Re \end{aligned}$$

since $u \in (Re)^\times$. Hence, by Lemma 3.1, we have $\dim_{\mathbb{F}_q}(R\mathbf{a}) = \dim_{\mathbb{F}_q}(Re) = \dim_{\mathbb{F}_q}(Re') = \dim_{\mathbb{F}_q}(R\mathbf{b})$. Therefore, $R\mathbf{b} = R\mathbf{a}$. \square

3.1 Enumeration of 1-Generator Quasi-Abelian Codes

First, we focus on the number of 1-generator H -quasi-abelian codes of a given idempotent generator. Then, the number of 1-generator H -quasi-abelian codes in $\mathbb{F}_q[G]$ follows.

Proposition 3.3. *Let $\{e_1, e_2, \dots, e_r\}$ be a set of primitive idempotents of R and $e = e_1 + e_2 + \dots + e_r$. Then the following statements hold.*

- i) e_1, e_2, \dots, e_r are pairwise orthogonal (non-zero) idempotents of Se .
- ii) e_j is the identity of Se_j for all $j = 1, 2, \dots, r$.
- iii) e is the identity of Se .
- iv) $Se = Se_1 \oplus Se_2 \oplus \dots \oplus Se_r$.

Proof. For i), it is clear that e_1, e_2, \dots, e_r are pairwise orthogonal (non-zero) idempotents in S . They are in Se since $e_j = e_j e \in Se$ for all $j = 1, 2, \dots, r$. The statements ii) and iii) follow since $se_j = se_j^2 = (se_j)e_j$ for all $se_j \in Se_j$ and $se = se^2 = (se)e$ for all $se \in Se$. The last statement can be verified using i). \square

Corollary 3.4. *Let $\{e_1, e_2, \dots, e_r\}$ be a set of primitive idempotents of R and $e = e_1 + e_2 + \dots + e_r$. Then the following statements hold.*

- i) e_1, e_2, \dots, e_r are pairwise orthogonal (non-zero) idempotents of Re .
- ii) e_j is the identity of Re_j for all $j = 1, 2, \dots, r$.
- iii) e is the identity of Re .
- iv) $Re = Re_1 \oplus Re_2 \oplus \dots \oplus Re_r$, where Re_j is isomorphic to an extension field of \mathbb{F}_q for all $j = 1, 2, \dots, r$.

$$\text{Let } \Omega = \left\{ \sum_{j=1}^r A_j \mid A_j \in (Se_j)^* \right\} \subset Se.$$

Lemma 3.5. *Let $A = \sum_{i=1}^l \alpha_i a_i \in S$, where $a_i \in R$, and let $b \in R$. Then $RA \subseteq Sb$ if and only if $Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Rb$.*

Proof. Assume that $RA \subseteq Sb$. Then $A = Bb$ for some $B \in S$. Write $B = \sum_{i=1}^l \alpha_i b_i$, where $b_i \in R$. Then $a_i = bb_i$ for all $i = 1, 2, \dots, l$. Hence, we have

$$\sum_{i=1}^l r_i a_i = \sum_{i=1}^l r_i b b_i = \left(\sum_{i=1}^l r_i b_i \right) b \in Rb.$$

for all $\sum_{i=1}^l r_i a_i \in Ra_1 + Ra_2 + \cdots + Ra_l$.

Conversely, it suffices to show that $A \in Sb$. Since $Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Rb$, we have $a_i \in Rb$ for all $i = 1, 2, \dots, l$. Then, for each $i = 1, 2, \dots, l$, there exists $r_i \in R$ such that $a_i = r_i b$. Hence,

$$A = \sum_{i=1}^l \alpha_i a_i = \sum_{i=1}^l \alpha_i r_i b = \left(\sum_{i=1}^l \alpha_i r_i \right) b \in Sb.$$

□

Lemma 3.6. *Let $A = \sum_{i=1}^l \alpha_i a_i \in Se$, where $a_i \in R$. Then $A \in \Omega$ if and only if $Re = Ra_1 + Ra_2 + \cdots + Ra_l$.*

Proof. First, we note that $RA \subseteq Se$ since $A \in Se$. Then $Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Re$ by Lemma 3.5.

Assume that $A \in \Omega$. Then $A = A_1 + A_2 + \cdots + A_r$, where $A_j \in (Se_j)^*$. We have $Ae_j = A_j \neq 0$ for all $j = 1, 2, \dots, r$. Suppose that $Ra_1 + Ra_2 + \cdots + Ra_l \subsetneq Re$. By Corollary 3.4, we have $Re = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$. Then

$$Ra_1 + Ra_2 + \cdots + Ra_l \subseteq \widehat{Re_j} = R(e - e_j)$$

for some $j \in \{1, 2, \dots, r\}$, where $\widehat{Re_j} = Re_1 \oplus \cdots \oplus Re_{j-1} \oplus Re_{j+1} \oplus \cdots \oplus Re_r$. By Lemma 3.5, we have

$$0 \neq A_j = Ae_j \in RA \subseteq S(e - e_j),$$

a contradiction. Therefore, $Ra_1 + Ra_2 + \cdots + Ra_l = Re$.

Conversely, assume that $Re = Ra_1 + Ra_2 + \cdots + Ra_l$. Then $RA \subseteq Se$ by Lemma 3.5. Since $A \in Se$, by Theorem 3.3, we have $A = A_1 + A_2 + \cdots + A_r$, where $A_j \in Se_j$ for all $j = 1, 2, \dots, r$. Suppose that $A_j = 0$ for some $j \in \{1, 2, \dots, r\}$. Then $RA = \widehat{RA_j} \subseteq \widehat{Se_j} = S(e - e_j)$. By Lemma 3.5, we have

$$Re = Ra_1 + Ra_2 + \cdots + Ra_l \subseteq R(e - e_j)$$

which is a contradiction. Hence, $A_j \in (Se_j)^*$ for all $j = 1, 2, \dots, r$. □

Corollary 3.7. *Let $A = \sum_{i=1}^l \alpha_i a_i \in Se_j$, where $a_i \in R$. Then $A \in (Se_j)^*$ if and only if $Re_j = Ra_1 + Ra_2 + \cdots + Ra_l$.*

Let k_j denote the \mathbb{F}_q -dimension of e_j . Then Re_j is isomorphic to a finite field of q^{k_j} elements.

For $j = 1, 2, \dots, r$, define an equivalence relation on $(Se_j)^*$ by

$$A \sim B \iff \exists u \in (Re_j)^\times \text{ such that } A = uB.$$

For $A \in (Se_j)^*$, denote by $[A]$ the equivalence class of A and let $[(Se_j)^*] = \{[A] \mid A \in (Se_j)^*\}$.

Lemma 3.8. For $A \in (Se_j)^*$, we have $|[A]| = q^{k_j} - 1$.

Proof. Define $\rho : (Re_j)^\times \rightarrow [A]$,

$$u \mapsto uA.$$

From the definition of \sim , ρ is a well-defined surjective map. For $u_1, u_2 \in (Re_j)^\times$, if $u_1A = u_2A$, then $(u_1 - u_2)A = 0$. Write $A = \sum_{i=1}^l \alpha_i a_i$, where $a_i \in R$. Then $a_i(u_1 - u_2) = 0$ for all $i = 1, 2, \dots, l$. Since $A \in (Se_j)^*$, by Corollary 3.7, we can write $e_j = \sum_{i=1}^i r_i a_i$, where $r_i \in R$. Hence,

$$e_j(u_1 - u_2) = \left(\sum_{i=1}^i r_i a_i \right) (u_1 - u_2) = \sum_{i=1}^i r_i a_i (u_1 - u_2) = 0 \in Re_j.$$

Since e_j is the identity of Re_j , it follows that $u_1 = u_2 \in (Re_j)^\times$. Therefore, ρ is a bijection. \square

Corollary 3.9. For each $i = 1, 2, \dots, r$, we have

$$|[(Se_j)^*]| = \frac{|(Se_j)^*|}{|[A]|} = \frac{q^{lk_j} - 1}{q^{k_j} - 1}.$$

Let $[\Omega] = \prod_{j=1}^r [(Se_j)^*]$. Then $|[\Omega]| = \prod_{j=1}^r \frac{q^{lk_j} - 1}{q^{k_j} - 1}$.

The number of 1-generator quasi-abelian codes sharing a idempotent has been determined in [8, Corollary 6.1]. Here, an alternative proof using a different technique is provided.

Theorem 3.10. Let \mathfrak{C} denote the set of all 1-generator H -quasi-abelian codes in $\mathbb{F}_q[G]$ with idempotent generator e . Then there exists a one-to-one correspondence between $[\Omega]$ and \mathfrak{C} . Hence, the number of 1-generator quasi-abelian codes having e as their idempotent generator is

$$\prod_{j=1}^r \frac{q^{lk_j} - 1}{q^{k_j} - 1}.$$

Proof. Define $\sigma : [\Omega] \rightarrow \mathfrak{C}$,

$$([A_1], [A_2], \dots, [A_r]) \mapsto R\mathbf{a},$$

where $A := A_1 + A_2 + \dots + A_r \in Se$ is viewed as $A = \sum_{i=1}^l \alpha_i a_i$ and $\mathbf{a} := (a_1, a_2, \dots, a_l)$.

Since $A_j \in (Se_j)^*$ for all $j = 1, 2, \dots, r$, we have $A \in \Omega$. Then $Re = Ra_1 + Ra_2 + \dots + Ra_l$ by Lemma 3.6, and hence, $R\mathbf{a}$ is a 1-generator quasi-abelian code with idempotent generator e , i.e., $R\mathbf{a} \in \mathfrak{C}$.

For $([A_1], [A_2], \dots, [A_r]) = ([B_1], [B_2], \dots, [B_r]) \in [\Omega]$, there exists $u_j \in (Re_j)^\times$ such that $A_j = u_j B_j$ for all $j = 1, 2, \dots, r$. Let $u := u_1 + u_2 + \dots + u_r$. Then

$$u(u_1^{-1} + u_2^{-1} + \dots + u_r^{-1}) = e_1 + e_2 + \dots + e_r = e$$

is the identity of Re (see Corollary 3.4), where u_j^{-1} refers to the inverse of u_j in Re_j . Hence, u is a unit in $(Re)^\times$. Let $A := \sum_{j=1}^r A_j$ and $B := \sum_{j=1}^r B_j$. Then

$$A = \sum_{j=1}^r u_j B_j = uB.$$

Hence, $R\mathbf{a} = R\mathbf{b}$ by Lemma 3.2. Therefore, σ is a well-defined map.

For $([A_1], [A_2], \dots, [A_r]), ([B_1], [B_2], \dots, [B_r]) \in [\Omega]$, if $R\mathbf{a} = R\mathbf{b}$, then, by Lemma 3.2, there exists $u \in (Re)^\times$ such that $A = uB$. Then $A_j = uB_j = ue_jB_j$ since e_j is the identity of Se_j by Proposition 3.3. Since $A_j \in (Se_j)^*$, ue_j is a non-zero in Re_j which is a finite field. Thus ue_j is a unit in $(Re_j)^\times$. Hence,

$$([A_1], [A_2], \dots, [A_r]) = ([B_1], [B_2], \dots, [B_r])$$

which implies that σ is an injective map.

To verify that σ is surjective, let $R\mathbf{a} \in \mathfrak{C}$, where $\mathbf{a} = (a_1, a_2, \dots, a_l)$. Then $Re = Ra_1 + Ra_2 + \dots + Ra_l$. Hence, by Lemma 3.6, we conclude that

$$A := \sum_{i=1}^l \alpha_i a_i \in \Omega.$$

Write $A = \sum_{j=1}^r A_j$, where $A_j \in (Se_j)^*$. Then $([A_1], [A_2], \dots, [A_r]) \in [\Omega]$, and hence,

$$\sigma([A_1], [A_2], \dots, [A_r]) = R\mathbf{a}.$$

□

3.2 The Generators for 1-Generator Quasi-Abelian Codes

In this subsection, we establish an algorithm to find all 1-generator H -quasi-abelian codes in $\mathbb{F}_q[G]$. Note that every idempotent in $R := \mathbb{F}_q[H]$ can be written as a sum of primitive idempotents in R . Hence, it is sufficient to study H -quasi-abelian codes for a given idempotent generator.

Let $e = e_1 + e_2 + \dots + e_r$ be an idempotent in R , where, for $j = 1, 2, \dots, r$, e_j is the primitive idempotent in R induced by a q -cyclotomic class $S_q(h_j)$ for some $h_j \in H$.

For each $j = 1, 2, \dots, r$, assume that e_j is decomposed as

$$e_j = e_{j1} + e_{j2} + \dots + e_{js_j},$$

where, for each $i = 1, 2, \dots, s_j$, e_{ji} is the primitive idempotent in S defined corresponding to a q^l -cyclotomic class $S_{q^l}(h_{ji})$ for some $h_{ji} \in S_q(h_j)$.

Note that all the elements in $S_q(h_j)$ have the same order. Hence, the q^l -cyclotomic classes $S_{q^l}(h_{ji})$, for $1 \leq i \leq s_j$, have the same size. Without loss of generality, we assume that e_{j1} is defined corresponding to $S_{q^l}(h_j)$. For each $j = 1, 2, \dots, r$, let k_j and d_j denote the \mathbb{F}_q -dimension of e_j and the \mathbb{F}_{q^l} -dimension of e_{j1} , respectively. Then k_j and d_j are the smallest positive integers such that

$$q^{k_j} \cdot h_j = h_j \text{ and } q^{ld_j} \cdot h_j = h_j.$$

Then $k_j \mid ld_j$ which implies that $\frac{k_j}{\gcd(l, k_j)} \mid d_j$. Since $q^{l \frac{k_j}{\gcd(l, k_j)}} \cdot h_j = q^{k_j \frac{l}{\gcd(l, k_j)}} \cdot h_j = h_j$, we have $d_j \mid \frac{k_j}{\gcd(l, k_j)}$. It follows that $d_j = \frac{k_j}{\gcd(l, k_j)}$. Hence, e_{ji} 's have the same q^l -size $d_j = \frac{k_j}{\gcd(l, k_j)}$ and $s_j = \gcd(l, k_j)$.

Using arguments similar to those in the proof of Proposition 3.3, we conclude the following result.

Proposition 3.11. Let $\{e_1, e_2, \dots, e_r\}$ be a set of primitive idempotents of R . Assume that $e_j = e_{j1} + e_{j2} + \dots + e_{js_j}$, where e_{ji} is a primitive idempotent in S for all $i = 1, 2, \dots, s_j$. Then the following statements hold.

- i) For $j = 1, 2, \dots, r$, the elements $e_{j1}, e_{j2}, \dots, e_{js_j}$ are pairwise orthogonal (non-zero) idempotents of Se_j .
- ii) e_{ji} is the identity of Se_{ji} for all $j = 1, 2, \dots, r$ and $i = 1, 2, \dots, s_j$.
- iii) $e_j = e_{j1} + e_{j2} + \dots + e_{js_j}$ is the identity of Se_j for all $j = 1, 2, \dots, r$.
- iv) For $j = 1, 2, \dots, r$, we have $Se_j = Se_{j1} \oplus Se_{j2} \oplus \dots \oplus Se_{js_j}$, where Se_{ji} is an extension field of \mathbb{F}_q of order q^{ld_j} for all $i = 1, 2, \dots, s_j$.

Theorem 3.12. Let $j \in \{1, 2, \dots, r\}$ be fixed. For $i = 1, 2, \dots, s_j$, let π_i be a primitive element of Se_{ji} , a finite field of q^{ld_j} elements. Let $L_j = \frac{q^{ld_j} - 1}{q^{k_j} - 1}$ and $T_j = \{\infty, 0, 1, 2, \dots, q^{ld_j} - 2\}$. Then the elements

$$\pi_t^{\nu_t} + \pi_{t+1}^{\nu_{t+1}} + \dots + \pi_{s_j}^{\nu_{s_j}}, \quad (3.1)$$

for all $1 \leq t \leq s_j$, $0 \leq \nu_t \leq L_j - 1$, and $\nu_{t+1}, \nu_{t+2}, \dots, \nu_{s_j} \in T_j$, are a complete set of representatives of $[(Se_j)^*]$. (By convention, $\pi_i^\infty = 0$.)

Proof. Note that the number of elements in (3.1) is

$$L_j q^{ld_j(s_j-1)} + L_j q^{ld_j(s_j-2)} + \dots + L_j = \frac{q^{lk_j} - 1}{q^{k_j} - 1} = |[(Se_j)^*]|.$$

Hence, it suffices to show that the elements in (3.1) are in different equivalence classes. Let

$$A = \pi_t^{\nu_t} + \pi_{t+1}^{\nu_{t+1}} + \dots + \pi_{s_j}^{\nu_{s_j}} \text{ and } B = \pi_x^{\mu_x} + \pi_{x+1}^{\mu_{x+1}} + \dots + \pi_{s_j}^{\mu_{s_j}},$$

where $0 \leq \nu_t, \mu_x \leq L_j - 1$, $\nu_{t+1}, \nu_{t+2}, \dots, \nu_{s_j} \in T_j$, and $\mu_{x+1}, \mu_{x+2}, \dots, \mu_{s_j} \in T_j$. Assume that $[A] = [B]$. Then there exists $u \in (Re_j)^\times$ such that

$$\pi_t^{\nu_t} + \pi_{t+1}^{\nu_{t+1}} + \dots + \pi_{s_j}^{\nu_{s_j}} = A = uB = u\pi_x^{\mu_x} + u\pi_{x+1}^{\mu_{x+1}} + \dots + u\pi_{s_j}^{\mu_{s_j}}.$$

Since $\pi_t^{\nu_t} \in (Se_{jt})^\times$ and $u\pi_x^{\mu_x} \in (Se_{jx})^\times$, by the decomposition in Proposition 3.11, $t = x$ and $\pi_t^{\nu_t} = u\pi_t^{\mu_t} \in Se_{jt}$. Then $ue_{jt} = \pi_t^{\nu_t - \mu_t}$. Since $u \in (Re_j)^\times$, we have $u^{q^{k_j} - 1} = e_j$, and hence, $e_{jt} = e_{jt}e_j = \pi_t^{(\nu_t - \mu_t)(q^{k_j} - 1)}$. Since $0 \leq \nu_t, \mu_t \leq L_j - 1$ and π_t has order $q^{ld_j} - 1$, we conclude that $\nu_t = \mu_t$. Hence, $ue_{jt} = e_{jt} = e_j e_{jt}$ which implies $(u - e_j)e_{jt} = 0$ in Se_{jt} . It follows that

$$S(u - e_j) \subseteq S(e_{j1} + \dots + e_{j,t-1} + e_{j,t+1} + \dots + e_{js_j}) \subsetneq Se_j.$$

Since $u, e_j \in Re_j$, $u - e_j \in Re_j$ and $R(u - e_j) \subsetneq Re_j$. Hence, $R(u - e_j)$ is the zero ideal, i.e., $u = e_j$. Therefore, $A = uB = e_j B = B$ since e_j is the identity of Se_j . \square

The following corollary now follows from Theorem 3.10 and Theorem 3.12.

Corollary 3.13. *Let $\{e_1, e_2, \dots, e_r\}$ be a set of primitive idempotents of R and $e = e_1 + e_2 + \dots + e_r$. Then all 1-generator quasi-abelian codes having e as their idempotent generator are of the form*

$$A_1 + A_2 + \dots + A_r,$$

where $A_j \in (Se_j)^*$ is as defined in (3.1).

Combining the results above, we summarize an algorithm to find all 1-generator H -quasi-abelian codes as in $\mathbb{F}_q[G]$ follows.

Algorithm

For abelian groups $H \leq G$ and a finite field \mathbb{F}_q with $\gcd(q, |H|) = 1$ and $[G : H] = l$, do the following steps.

1. Compute the q -cyclotomic classes of H in G .
2. Compute the set $\{e_1, e_2, \dots, e_r\}$ of primitive idempotents of $R = \mathbb{F}_q[H]$ (see [5, Proposition II.4]).
3. For each $1 \leq j \leq r$, compute a set B_j of a complete set of representatives of $[(Se_j)^*]$ (see Theorem 3.12).
4. Compute the idempotents of R , i.e., the set

$$T = \left\{ \sum_{j=1}^t e_{i_j} \mid 1 \leq t \leq r \text{ and } 1 \leq i_1 < i_2 < \dots < i_t \leq r \right\}.$$

5. For each $e = \sum_{j=1}^t e_{i_j} \in T$, compute the 1-generator quasi-abelian codes having e as their idempotent generator of the form

$$A_1 + A_2 + \dots + A_t,$$

where $A_j \in B_{i_j}$ (see Corollary 3.13).

6. Run e over all elements of T . Then the 1-generator H -quasi-abelian codes in $\mathbb{F}_q[G]$ are obtained.
-

Example 3.14. *Let $q = 2$, $G = \mathbb{Z}_3 \times \mathbb{Z}_6$ and $H = \mathbb{Z}_3 \times 2\mathbb{Z}_6$. Denote by $a_0 := (0, 0)$, $a_1 := (1, 0)$, $a_2 := (2, 0)$, $a_3 := (0, 2)$, $a_4 := (1, 2)$, $a_5 := (2, 2)$, $a_6 := (0, 4)$, $a_7 := (1, 4)$, and $a_8 := (2, 4)$, the elements in H . Then $l = [G : H] = 2$ and the elements in H can be partitioned into the following 2-cyclotomic classes $S_2(a_0) = \{a_0\}$, $S_2(a_1) = \{a_1, a_2\}$, $S_2(a_3) = \{a_3, a_6\}$, $S_2(a_4) = \{a_4, a_8\}$, and $S_2(a_5) = \{a_5, a_7\}$. From [5, Proposition II.4], we note that*

$$\begin{aligned} e_1 &= Y^{a_0} + Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8}, \\ e_2 &= Y^{a_1} + Y^{a_2} + Y^{a_4} + Y^{a_5} + Y^{a_7} + Y^{a_8}, \\ e_3 &= Y^{a_3} + Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8}, \\ e_4 &= Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_4} + Y^{a_6} + Y^{a_8}, \\ e_5 &= Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_5} + Y^{a_6} + Y^{a_7} \end{aligned}$$

are primitive idempotents of $R := \mathbb{F}_2[H]$ induced by $S_2(a_0)$, $S_2(a_1)$, $S_2(a_3)$, $S_2(a_4)$, and $S_2(a_5)$, respectively.

Let $e := e_1 + e_2 + e_3$. It follows from Theorem 3.10 that the number of 1-generator H -quasi abelian codes in $\mathbb{F}_2[G]$ with idempotent generator e is $3 \cdot 5 \cdot 5 = 75$.

Let $S := \mathbb{F}_4[H]$, where $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2 = 1 + \alpha\}$. Then $e_2 = e_{21} + e_{22}$ and $e_3 = e_{31} + e_{32}$, where

$$\begin{aligned} e_{21} &= Y^{a_0} + \alpha^2 Y^{a_1} + \alpha Y^{a_2} + Y^{a_3} + \alpha^2 Y^{a_4} + \alpha Y^{a_5} + Y^{a_6} + \alpha^2 Y^{a_7} + \alpha Y^{a_8}, \\ e_{22} &= Y^{a_0} + \alpha Y^{a_1} + \alpha^2 Y^{a_2} + Y^{a_3} + \alpha Y^{a_4} + \alpha^2 Y^{a_5} + 1Y^{a_6} + \alpha Y^{a_7} + \alpha^2 Y^{a_8}, \\ e_{31} &= Y^{a_0} + Y^{a_1} + Y^{a_2} + \alpha^2 Y^{a_3} + \alpha^2 Y^{a_4} + \alpha^2 Y^{a_5} + \alpha Y^{a_6} + \alpha Y^{a_7} + \alpha Y^{a_8}, \\ e_{32} &= Y^{a_0} + Y^{a_1} + Y^{a_2} + \alpha Y^{a_3} + \alpha Y^{a_4} + \alpha Y^{a_5} + \alpha^2 Y^{a_6} + \alpha^2 Y^{a_7} + \alpha^2 Y^{a_8} \end{aligned}$$

are primitive idempotents in S induced by 4-cyclotomic classes $\{a_1\}$, $\{a_2\}$, $\{a_3\}$ and $\{a_6\}$, respectively.

Now, we have $k_1 = 1$, $k_2 = k_3 = 2$, $d_1 = d_2 = d_3 = 1$, $s_1 = 1$, and $s_2 = s_3 = 2$. It follows that $L_1 = \frac{2^2-1}{2-1} = 3$, $L_2 = L_3 = \frac{2^2-1}{2^2-1} = 1$, and $T_1 = T_2 = T_3 = \{\infty, 0, 1, 2\}$.

Then αe_1 , αe_{21} , αe_{22} , αe_{31} , and αe_{32} are primitive elements of Se_1 , Se_{21} , Se_{22} , Se_{31} , and Se_{32} , respectively. Therefore, we have that

$$\begin{aligned} B_1 &= \{e_1, \alpha e_1, \alpha^2 e_1\}, \\ B_2 &= \{e_{21}, e_{21} + e_{22}, e_{21} + \alpha e_{22}, e_{21} + \alpha^2 e_{22}, e_{22}\}, \text{ and} \\ B_3 &= \{e_{31}, e_{31} + e_{32}, e_{31} + \alpha e_{32}, e_{31} + \alpha^2 e_{32}, e_{32}\} \end{aligned}$$

are complete sets of representatives of $[(Se_1)^*]$, $[(Se_2)^*]$, and $[(Se_3)^*]$, respectively. Hence, all the generators of the 75 1-generator H -quasi abelian codes in $\mathbb{F}_2[G]$ with idempotent generator e are of the form

$$A_1 + A_2 + A_3,$$

where $A_i \in B_i$ for all $i = 1, 2, 3$.

In order to find permutation inequivalent 1-generator H -quasi abelian codes, the following theorem is useful.

Theorem 3.15. Let $H \leq G$ be finite abelian groups of index $[G : H] = l$ and let $\{\alpha^{q^i} \mid 1 \leq i \leq l\}$ be a fixed basis of \mathbb{F}_{q^l} over \mathbb{F}_q . If $A = \sum_{i=1}^l a_i \alpha^{q^i} \in Se$, then A and $A^q = \sum_{i=1}^l a_i^q \alpha^{q^{i+1}}$ generate permutation equivalent H -quasi abelian codes (viewed in $\mathbb{F}_q[G]$) with the same idempotent generator.

Proof. Let e be the idempotent generator of a quasi-abelian code RA . Then

$$Ra_1^q + Ra_2^q + \cdots + Ra_l^q \subseteq Ra_1 + Ra_2 + \cdots + Ra_l = Re$$

Assume that $e = \sum_{i=1}^l r_i a_i$, where $r_i \in R$. It follows that

$$e = e^q = \sum_{i=1}^l r_i^q a_i^q \in Ra_1^q + Ra_2^q + \cdots + Ra_l^q.$$

Hence, $Re = Ra_1^q + Ra_2^q + \cdots + Ra_l^q$. Therefore, A and A^q generate codes with the same idempotent generator e .

Let $\psi : R \rightarrow R$ be a ring homomorphism defined by

$$\gamma \mapsto \gamma^q.$$

Let $\gamma = \sum_{h \in H} \gamma_h Y^h$ and $\beta = \sum_{h \in H} \beta_h Y^h$ be elements in R , where γ_h and β_h are elements in \mathbb{F}_q . If $\psi(\gamma) = \psi(\beta)$, then

$$0 = \gamma^q - \beta^q = (\gamma - \beta)^q = \sum_{h \in H} (\gamma_h - \beta_h) Y^{q \cdot h}.$$

By comparing the coefficients, we have $\gamma_h = \beta_h$ for all $h \in H$, i.e., $\gamma = \beta$. Hence, ψ is a ring automorphism and

$$\begin{aligned} R(a_l^q, a_1^q, \dots, a_{l-1}^q) &= R(\psi(a_l), \psi(a_1), \dots, \psi(a_{l-1})) \\ &= \Psi(R(a_l, a_1, \dots, a_{l-1})), \end{aligned} \tag{3.2}$$

where Ψ is a natural extension of ψ to R^l .

Since $\psi(\gamma) = \sum_{h \in H} \gamma_h Y^{q \cdot h}$, $\psi(\gamma)$ is just a permutation on the coefficients of γ . Hence, by (3.2), $\Psi \circ \Phi$ is a permutation on $\mathbb{F}_q[G]$ such that $\Phi^{-1}(R(a_l^q, a_1^q, \dots, a_{l-1}^q))$ is permutation equivalent to $\Phi^{-1}(R(a_l, a_1, \dots, a_{l-1}))$ in $\mathbb{F}[G]$, where Φ is the R -module isomorphism defined in (2.1). Therefore, the result follows since $R(a_l, a_1, \dots, a_{l-1})$ is permutation equivalent to $R(a_1, a_2, \dots, a_l)$. \square

4 Computational Results

It has been shown in [7] and [8] that a family of quasi-abelian codes contains various new and optimal codes. Here, we present other 2 new codes from the quasi-abelian codes together with 1 new code obtained by shortening of one of these codes.

Given an abelian group $H = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ of order $n = n_1 n_2$, denote by $u = (u_0, u_1, u_2, \dots, u_{n-1}) \in \mathbb{F}_q^n$ the vector representation of

$$u = \sum_{j=0}^{n_2-1} \sum_{i=0}^{n_1-1} u_{jn_1+i} Y^{(i,j)} \text{ in } \mathbb{F}_q[H].$$

Let

$$C_{(a,b)} := \{(fa, fb) \mid f \in \mathbb{F}_q[H]\}, \tag{4.1}$$

where a and b are elements in $\mathbb{F}_q[H]$. Using (4.1), 2 quasi-abelian codes whose minimum distance improves on Grassl's online table [6] can be found. The codes C_1 and C_2 are

presented in Table 1 and the generator matrices of C_1 and C_2 are

$$G_1 = \begin{bmatrix} I_{14} & \begin{bmatrix} 1 & 3 & 0 & 3 & 4 & 1 & 3 & 2 & 0 & 4 & 1 & 2 & 1 & 4 & 0 & 4 & 1 & 0 & 4 & 3 & 0 & 4 \\ 1 & 3 & 4 & 4 & 3 & 1 & 4 & 0 & 2 & 4 & 1 & 3 & 0 & 2 & 2 & 4 & 3 & 1 & 1 & 3 & 4 & 0 \\ 1 & 4 & 4 & 3 & 4 & 0 & 4 & 0 & 0 & 1 & 0 & 3 & 1 & 2 & 0 & 1 & 0 & 3 & 2 & 4 & 4 & 4 \\ 4 & 4 & 3 & 3 & 4 & 2 & 3 & 3 & 1 & 3 & 4 & 0 & 3 & 3 & 2 & 1 & 1 & 1 & 1 & 0 & 3 & 0 \\ 4 & 3 & 3 & 4 & 3 & 2 & 4 & 2 & 3 & 2 & 3 & 2 & 2 & 3 & 0 & 3 & 2 & 1 & 0 & 1 & 4 & 3 \\ 4 & 4 & 2 & 4 & 4 & 1 & 4 & 1 & 2 & 4 & 2 & 1 & 4 & 0 & 0 & 1 & 1 & 2 & 0 & 4 & 0 & 4 \\ 0 & 2 & 1 & 1 & 3 & 1 & 4 & 1 & 1 & 2 & 1 & 0 & 1 & 1 & 4 & 2 & 0 & 0 & 1 & 3 & 2 & 3 \\ 0 & 1 & 2 & 1 & 4 & 3 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 4 & 1 & 0 & 0 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 1 & 4 & 3 & 1 & 2 & 1 & 0 & 1 & 1 & 4 & 2 & 1 & 0 & 1 & 0 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 3 & 4 & 4 & 4 & 4 & 1 & 3 & 1 & 4 & 4 & 3 & 3 & 1 & 0 & 1 & 2 & 2 & 4 \\ 1 & 2 & 3 & 1 & 4 & 0 & 2 & 2 & 4 & 3 & 4 & 0 & 4 & 1 & 2 & 2 & 0 & 1 & 1 & 3 & 3 & 2 \\ 1 & 1 & 3 & 2 & 2 & 1 & 3 & 4 & 2 & 3 & 4 & 1 & 3 & 0 & 4 & 1 & 0 & 0 & 2 & 1 & 4 & 3 \\ 4 & 0 & 4 & 1 & 0 & 3 & 2 & 4 & 0 & 1 & 0 & 3 & 2 & 2 & 2 & 1 & 1 & 0 & 4 & 1 & 4 & 0 \\ 4 & 1 & 4 & 0 & 2 & 3 & 0 & 0 & 4 & 1 & 2 & 3 & 0 & 3 & 4 & 3 & 0 & 1 & 4 & 1 & 0 & 4 \end{bmatrix} \end{bmatrix}$$

and

$$G_2 = \begin{bmatrix} I_{11} & \begin{bmatrix} 0 & 1 & 0 & 4 & 4 & 0 & 0 & 1 & 4 & 4 & 0 & 4 & 1 & 3 & 2 & 3 & 3 & 1 & 1 & 3 & 3 & 2 & 0 & 1 & 4 \\ 4 & 4 & 1 & 1 & 2 & 1 & 2 & 4 & 1 & 4 & 3 & 2 & 1 & 4 & 4 & 3 & 2 & 4 & 2 & 0 & 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 4 & 0 & 0 & 0 & 4 & 4 & 4 & 1 & 4 & 1 & 0 & 2 & 3 & 3 & 1 & 1 & 3 & 3 & 2 & 3 & 1 & 4 & 0 \\ 0 & 1 & 0 & 0 & 4 & 0 & 4 & 1 & 0 & 3 & 1 & 3 & 0 & 3 & 1 & 4 & 1 & 3 & 4 & 1 & 4 & 3 & 3 & 4 & 1 \\ 4 & 4 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 3 & 3 & 4 & 1 & 4 & 3 & 1 & 4 & 1 & 3 & 0 & 3 & 1 & 3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 3 & 1 & 3 & 0 & 1 & 1 & 4 & 3 & 3 & 4 & 1 & 4 & 3 & 1 & 4 & 1 & 3 \\ 1 & 1 & 4 & 0 & 4 & 0 & 4 & 3 & 2 & 1 & 0 & 0 & 4 & 1 & 3 & 1 & 2 & 3 & 3 & 2 & 3 & 4 & 2 & 4 & 2 \\ 4 & 0 & 0 & 4 & 0 & 0 & 1 & 4 & 1 & 0 & 2 & 3 & 3 & 1 & 1 & 3 & 3 & 2 & 3 & 1 & 4 & 0 & 4 & 4 & 1 \\ 0 & 4 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 3 & 2 & 1 & 2 & 4 & 2 & 2 & 4 & 4 & 3 & 1 & 2 & 0 & 0 & 3 & 3 \\ 1 & 1 & 0 & 0 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 4 & 4 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 \end{bmatrix} \end{bmatrix},$$

respectively.

Table 1: New Codes from Quasi-Abelian Codes

name	$C_{(a,b)}$	H	a, b
$C1$	$[36, 14, 15]_5$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$a = (3, 3, 3, 0, 0, 1, 4, 3, 4, 0, 4, 4, 4, 4, 3, 0, 1, 0)$ $b = (2, 4, 1, 1, 3, 3, 0, 0, 4, 4, 1, 0, 0, 1, 4, 2, 2, 4)$
$C2$	$[36, 11, 18]_5$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$a = (2, 4, 4, 3, 4, 4, 3, 2, 4, 3, 4, 4, 3, 4, 2, 3, 4, 4)$ $b = (3, 0, 0, 0, 3, 3, 3, 0, 3, 0, 3, 0, 3, 0, 1, 1, 1, 1)$

By puncturing C_2 at the first coordinate, a $[35, 11, 17]_5$ code can be obtained with minimum distance improved by 1 from Grassl's online table [6]. All the computations are done using MAGMA [3].

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